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# ANALYTICAL METHOD FOR PREDICTION OF THE STRENGTH OF COMPOSITE LAMINATES*

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The problem of prediction of the strength of composite laminates based on the properties of the individual laminae and the structure of the composite has been solved for the general case in [1], where parametric equations were obtained for the limiting surface for a composite of arbitrary structure. However, application of these equations for practical computations requires a computer.

In this paper, we propose a technique allowing us to solve the considered problem by a simpler method for a composite with a structure in which there are identical layers with orientation angles $(+\varphi_j)$ and $(-\varphi_j)$, where $j = \overline{1, n}$, n is the number of layers. There are not other constraints on the structure surface of a composite of structure $[\pm\varphi]_c$ using the hypothesis that the strain rate field is uniform over the thickness of the packet of layers. The theoretical results are compared with the corresponding experimental results of other authors. For prediction of the strength of a composite of arbitrary structure, we have used the Voigt model [2] in the theory of mixtures. As an example, we solve the problem of the supporting capacity of the composite tube of the drive shaft of an automobile.

1. The limiting surface in the space of the stresses $\sigma_{ij}$ for an orthotropic monolayer in many cases of practical importance may be described by a second-degree equation [Refs. 3, 4, and others]:

$$a\sigma_{xx}^2 + 2b\sigma_{xx}\sigma_{yy} + c\sigma_{yy}^2 + 2d\sigma_{xx} + 2e\sigma_{yy} + l\sigma_{xy}^2 + m\sigma_{xz}^2 + n\sigma_{yz}^2 = 1. \tag{1.1}$$

Here xyz is a coordinate system whose axes coincide with the axes of the orthotropy of the monolayer (the z axis is orthogonal to the plane of the layer). The coefficients $a,\ldots,n$ of Eq. (1.1) are defined in terms of the strength characteristics of the monolayer.

Let us introduce the coordinate system $\xi_1\xi_2z$ connected with an element of the structure. Let us consider two jointly functioning identical monolayers whose strength properties are determined by Eq. (1.1). One of the layers makes the angle $(+\varphi)$ with the $\xi_1$ axis, the other layer makes the angle $(-\varphi)$ with the same axis. In the system $\xi_1\xi_2z$, Eq. (1.1) for the layer with orientation angle $(+\varphi)$ has the form

$$\begin{aligned}\Phi^+ \equiv A(\sigma_{11}^+)^2 + 2B\sigma_{11}^+\sigma_{22}^+ + C(\sigma_{22}^+)^2 + 2D\sigma_{11}^+ + 2E\sigma_{22}^+ + L(\sigma_{12}^+)^2 + \\ + 2P\sigma_{11}^+\sigma_{12}^+ + 2R\sigma_{22}^+\sigma_{12}^+ + 2Q\sigma_{12}^+ + \\ + K(\sigma_{31}^+)^2 + 2M\sigma_{31}^+\sigma_{32}^+ + N(\sigma_{32}^+)^2 = 1,\end{aligned} \tag{1.2}$$

and for the layer with orientation angle $(-\varphi)$, it has the form

$$\Phi^- \equiv A(\sigma_{11}^-)^2 + 2B\sigma_{11}^-\sigma_{22}^- + C(\sigma_{22}^-)^2 + 2D\sigma_{11}^- + 2E\sigma_{22}^- + L(\sigma_{12}^-)^2 -$$

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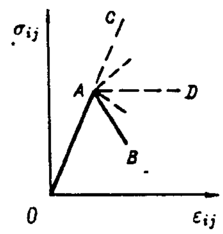


Fig. 1

$$
\begin{aligned}
&-2P\sigma_{11}^{-}\sigma_{12}^{-}-2R\sigma_{22}^{-}\sigma_{12}^{-}-2Q\sigma_{12}^{-}+\\
&\quad+K(\sigma_{31}^{-})^{2}-2M\sigma_{31}^{-}\sigma_{32}^{-}+N(\sigma_{32}^{-})^{2}=1,
\end{aligned}
\tag{1.3}
$$

where the coefficients A,.., N depend linearly on the coefficients $a,\ldots,$n of Eq. (1.1) and are functions of the angle $\varphi$.

In Fig. 1, we present the conventional stress $\sigma_{ij}$ – strain $\varepsilon_{ij}$ diagram (the broken line OAB). The section OA corresponds to the stable state of the material, while the section AB corresponds to the unstable state of the material. We assume that transition from the stable state to the unstable state occurs directly in the time interval Δt. The material in this time experiences the full spectrum of states. We will consider as the virtual $\sigma_{ij}-\varepsilon_{ij}$ diagram the one which would occur if we could stabilize the properties of the material at the considered moment of time. The virtual $\sigma_{ij}-\varepsilon_{ij}$ diagrams are located at the limits of the angle CAB (Fig. 1, dashed lines).

Among the virtual diagrams there is one which is parallel to the $\varepsilon_{ij}$ axis (the line AD). The state of the material corresponding to the diagram AD we take as the limiting state. In the limiting state, the material is stable (the Drucker postulate is valid [5]), i.e., $\delta\sigma_{ij}\delta\varepsilon_{ij} \geq 0$. Here the $\delta\sigma_{ij}$ are infinitesimally small stress increments, the $\delta\varepsilon_{ij}$ are infinitesimally small strain increments. As the loading surface in the limiting state, we take the strength surface of the considered material, described by the equation $\Phi(\sigma_{ij}) = 0$. The "vector" of the increments $\delta\sigma_{ij}$ is directed along the tangent to the loading surface. In other words, during fracture the material passes through a state for which the relations of the associative law for strain are valid

$$
\delta\varepsilon_{ij}=\delta\lambda\frac{\partial\Phi}{\partial\sigma_{ij}}
\tag{1.4}
$$

for the condition that as the loading surface $\Phi(\sigma_{ij}) = 0$ in this state we take the strength surface for the material. In (1.4), $\delta\lambda(\delta\varepsilon_{ij})$ is a scalar function. Dividing both sides of (1.4) by dt, we can go to the strain rates. Let us assume that in the limiting state, the strains are small.

Let us consider the case of a plane stress–strain state of two jointly functioning layers of orientation ($\pm\varphi$):

$$
\begin{gathered}
\dot\varepsilon_{11}^{+}=\dot\varepsilon_{11}^{-},\qquad \dot\varepsilon_{22}^{+}=\dot\varepsilon_{22}^{-},\qquad \dot\varepsilon_{12}^{+}=\dot\varepsilon_{12}^{-},\\
\sigma_{11}=0{,}5(\sigma_{11}^{+}+\sigma_{11}^{-}),\ \sigma_{22}=0{,}5(\sigma_{22}^{+}+\sigma_{22}^{-}),\ \sigma_{12}=0{,}5(\sigma_{12}^{+}+\sigma_{12}^{-}).
\end{gathered}
\tag{1.5}
$$

Using (1.2)-(1.4), let us write out the kinematic relations from (1.5):

$$
\begin{aligned}
&\dot\lambda^{+}(A\sigma_{11}^{+}+B\sigma_{22}^{+}+P\sigma_{12}^{+}+D)=\dot\lambda^{-}(A\sigma_{11}^{-}+B\sigma_{22}^{-}-P\sigma_{12}^{-}+D),\\
&\dot\lambda^{+}(B\sigma_{11}^{+}+C\sigma_{22}^{+}+R\sigma_{12}^{+}+E)=\dot\lambda^{-}(B\sigma_{11}^{-}+C\sigma_{22}^{-}-R\sigma_{12}^{-}+E),\\
&\dot\lambda^{+}(P\sigma_{11}^{+}+R\sigma_{22}^{+}+L\sigma_{12}^{+}+Q)=\dot\lambda^{-}(-P\sigma_{11}^{-}-R\sigma_{22}^{-}+L\sigma_{12}^{-}-Q)
\end{aligned}
\tag{1.6}
$$

Using the results in [1], for a plane stress–strain state of the layers we can write

$$
\dot\lambda^{2}=\delta_{11}\dot\varepsilon_{11}^{2}+\delta_{22}\dot\varepsilon_{22}^{2}+\delta_{33}\dot\varepsilon_{12}^{2}+2(\delta_{12}\dot\varepsilon_{11}\dot\varepsilon_{22}+\delta_{23}\dot\varepsilon_{22}\dot\varepsilon_{12}+\delta_{31}\dot\varepsilon_{12}\dot\varepsilon_{11}).
\tag{1.7}
$$

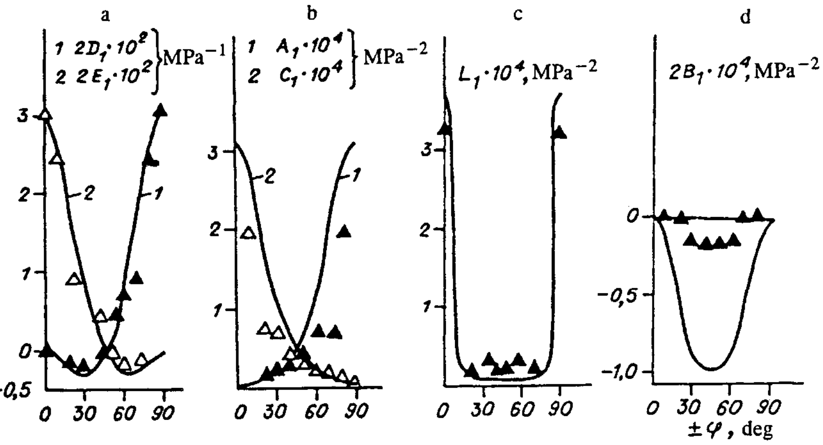


Fig. 2

Here $\delta_{ik}$ (i, k = $\overline{1, 3}$) is the algebraic cofactor of the (i, k)-th element of the determinant $\Delta$ [1]; based on Eqs. (1.2) and (1.3), we obtain

$$\Delta^+ = \begin{vmatrix} A & B & P \\ B & C & R \\ P & R & L \end{vmatrix}, \qquad \Delta^- = \begin{vmatrix} A & B & -P \\ B & C & -R \\ -P & -R & L \end{vmatrix}. \tag{1.8}$$

From (1.5), (1.7), (1.8) it follows that for the considered case of the stress–strain state of a composite of structure [$\pm\varphi$], the equality $\dot\lambda^+ = \dot\lambda^-$ holds. Then, using (1.6) and the statics relations from (1.5), we find

$$\begin{aligned} \sigma_{11}^+ &= \sigma_{11} + [(BR - CP)/(AC - B^2)]\sigma_{12}, \\ \sigma_{22}^+ &= \sigma_{22} + [(BP - AR)/(AC - B^2)]\sigma_{12}, \\ \sigma_{12}^+ &= \sigma_{12} - (P\sigma_{11} + R\sigma_{22} + Q)/L. \end{aligned} \tag{1.9}$$

Let us consider the case of a purely transverse shear of two jointly functioning layers of orientation ($\pm\varphi$):

$$\begin{gathered} \dot\varepsilon_{31}^+ = \dot\varepsilon_{31}^-, \qquad \dot\varepsilon_{32}^+ = \dot\varepsilon_{32}^-, \\ \sigma_{31} = 0{,}5(\sigma_{31}^+ + \sigma_{31}^-), \qquad \sigma_{32} = 0{,}5(\sigma_{32}^+ + \sigma_{32}^-). \end{gathered} \tag{1.10}$$

For the state of the material defined by relations (1.10), it follows that $\dot\lambda^+ = \dot\lambda^-$. Considering this equality and using the relations (1.2)-(1.4), (1.10), we obtain

$$\sigma_{31}^+ = \sigma_{31} - \frac{M}{K}\sigma_{32}, \quad \sigma_{32}^+ = \sigma_{32} - \frac{M}{N}\sigma_{31}. \tag{1.11}$$

Let us assume that in the limiting state in the general case of the stress–strain state of two jointly functioning layers of orientation ($\pm\varphi$), a combination of relations (1.5) and (1.10) holds; then, substituting (1.9) and (1.11) into (1.2), we find

$$\begin{aligned} &\frac{AL - P^2}{L}\sigma_{11}^2 + 2\frac{BL - PR}{L}\sigma_{11}\sigma_{22} + \frac{CL - R^2}{L}\sigma_{22}^2 + 2\frac{DL - PQ}{L}\sigma_{11} + \\ &+2\frac{EL - RQ}{L}\sigma_{22} + \Big(L + \frac{2BPR - CP^2 - AR^2}{AC - B^2}\Big)\sigma_{12}^2 + \\ &+\Big(1 - \frac{M^2}{KN}\Big)(K\sigma_{31}^2 + 2M\sigma_{31}\sigma_{32} + N\sigma_{32}^2) - \frac{Q^2}{L} = 1. \end{aligned} \tag{1.12}$$

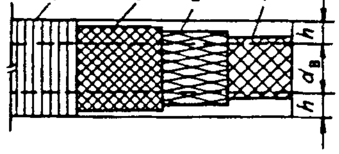


Fig. 3

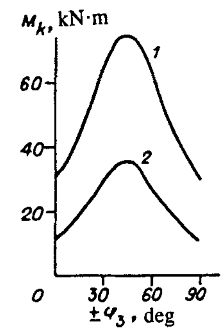


Fig. 4

Equation (1.12) is the equation for the limiting surface in the system $\xi_1\xi_2 z$ for a composite of structure $[+\varphi/-\varphi]_c$ (we 'can find substantiation for this claim, for example, in [6]). We should note that, expressing $\bar{\sigma}_{11}$, $\bar{\sigma}_{22}$, $\bar{\sigma}_{12}$, $\bar{\sigma}_{31}$, $\bar{\sigma}_{32}$ in terms of $\sigma_{11}$, $\sigma_{22}$, $\sigma_{12}$, $\sigma_{31}$, $\sigma_{32}$ and substituting these expressions into (1.3), we again obtain Eq. (1.12).

Let us rewrite (1.12) in the form

$$\begin{gathered} A_1\sigma_{11}^2 + 2B_1\sigma_{11}\sigma_{22} + C_1\sigma_{22}^2 + 2D_1\sigma_{11} + 2E_1\sigma_{22} + \\ + L_1\sigma_{12}^2 + K_1\sigma_{31}^2 + 2M_1\sigma_{31}\sigma_{32} + N_1\sigma_{32}^2 = 1, \end{gathered} \tag{1.13}$$

where

$$A_1 = \frac{AL - P^2}{L + Q^2}, \ldots; \qquad N_1 = \frac{L(KN - M^2)}{K(L + Q^2)}.$$

Determining the points of intersection of the surface (1.13) with the axes $\sigma_{11}$, $\sigma_{22}$, $\sigma_{12}$, $\sigma_{31}$, $\sigma_{32}$, we find the corresponding strengths of the composite of structure $[+\varphi/-\varphi]_c$:

$$\begin{gathered} \sigma_{11}^{ut} = \frac{-D_1 + \sqrt{D_1^2 + A_1}}{A_1}, \\ \sigma_{11}^{uc} = \frac{-D_1 - \sqrt{D_1^2 + A_1}}{A_1}, \ldots, \sigma_{32}^{u} = \pm\frac{1}{\sqrt{N_1}}. \end{gathered} \tag{1.14}$$

Here the index u means the limiting value of the corresponding stress; the index t refers to tensile stresses while the index c refers to compressive stresses.

In Fig. 2, we present the results of comparison of the values of the coefficients of Eq. (1.13), determined using the technique outlined above (solid lines), with the corresponding experimental values from [7] (light and dark triangles). As we see the qualitative agreement between the results is good. Qualitative discrepancies are connected primarily with the indication of fracture used in the experiment and with deviations from the geometric hypotheses assumed in deriving Eq. (1.12).

The results obtained can be used to predict the strength of hybrid composites when there are several stipulations. For plane strain of such composites, applying internal elementary forces to some surface $S_0$, in the general case we obtain internal moments in addition to internal forces. Let us assume that the stresses from the moments do not significantly affect the strength characteristics of the hybrid composite relative to tension compression, and shear over a rather broad range of variation of these moments. Then the strength properties of the composite can be predicted on the basis of the rule of mixtures

$$\sigma_{ij}^{u} = \sum_{k=1}^{n} \tilde{h}_k \sigma_{ij}^{u(k)},$$

where $\sigma_{ij}{}^{u}$ is the strength characteristic of the composite; $\sigma_{ij}^{u(k)}$ is the corresponding strength of two jointly functioning identical monolayers with stacking angles $\pm\varphi_k$, defined according to (1.4); $\tilde{h}_k = h_k/h$ is the relative thickness of two such layers; h is the thickness of the packet of layers.

**2.** Let us consider the problem of determining the limiting torque for the composite tube of the drive shaft of an automobile. The strength calculation for static action of an external torque is one of the necessary elements in the full calculation for designing such shafts. One possible structural variant for the drive shaft is shown in Fig. 3, where 1 is two internal spiral-crossed layers of glass-fiber reinforced plastic with fiber orientation angles $\pm 45°$, 2 are layers of glass-fiber reinforced plastic with fiber orientation angles $\pm 10°$, 3 are layers of high strength carbon-fiber reinforced plastic with fiber orientation angles $\pm\varphi_3$, 4 is a protective layer of glass-fiber reinforced plastic with fiber orientation angle 90° (the fiber orientation angle is measured from the generatrix of the cylinder). A composite tube of analogous structure has been successfully used in drive shafts of passenger cars [8].

As a rule, the external dimensions of the shaft are limited by design requirements. Below as an example we investigated the dependence of the supporting capacity of a composite tube on the values of the orientation angles $\pm\varphi_3$ of carbon-fiber reinforced plastic layers for two variants of its inner diameter and wall thickness. In Fig. 4, the limiting curve 1 corresponds to the following geometric parameters of the tube and the composite: $d_{in}$ = 120 mm (inner diameter of the tube), h = 8 mm (wall thickness), $h_1$ = 1.2 mm, $h_2$ = 3.6 mm, $h_3$ = 2.6 mm; $h_4$ = 0.6 mm (thickness of the layers); and curve 2 corresponds to the parameters: $d_{in}$ = 94 mm, h = 4 mm, $h_1$ = 1.4 mm, $h_2$ = 0, $h_3$ = 2.4 mm, $h_4$ = 0.4 mm. The strength characteristics of the glass-fiber reinforced plastic monolayer are taken from [9], and the strength characteristics of the carbon-fiber reinforced plastic monolayer are taken from [10].

Having available a set of curves similar to curves 1, 2 in Fig. 4, the designer can easily select the orientational variant in designing the drive shaft, which may be used as the basis for other types of calculation in the design process. In this case, we need to strive for a decrease in the overall mass of the shaft and the relative volume of the more expensive carbon-fiber reinforced plastic within the composite, simultaneously satisfying the conditions for strength, rigidity, durability, and other requirements imposed on the service conditions for the article.

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